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# DEVELOPMENT OF THE LUNAR AND SOLAR PERTURBATIONS IN THE MOTION OF AN ARTIFICIAL SATELLITE

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SUMMARY

Problems relating to the influence of lunar and solar perturbations on the motion of artificial satellites are analyzed by an extension of Cayley's development of the perturbative function in the lunar theory. In addition, the results are modified for incorporation into the Hansen-type theory used by the NASA Space Computing Center. The theory is applied to the orbits of the Vanguard I and Explorer VI satellites, and the results of detailed computations for these satellites are given together with a physical description of the perturbations in terms of resonance effects.

CONTENTS

# DEVELOPMENT OF THE LUNAR AND SOLAR PERTURBATIONS IN THE MOTION OF AN ARTIFICIAL SATELLITE*

## INTRODUCTION

This paper is concerned primarily with problems related to the influence of lunar and solar perturbations on the motion of artificial satellites. The importance of these problems is indicated by Kozai's discovery that the perigee height and lifetime of a satellite may be strongly affected by these perturbations.

The basis for the computations is provided by an analytical development of the disturbing function, which is an extension of Cayley's development of the solar perturbative function in the lunar theory. The relations between perigee-height variations and launch conditions have been investigated by using a modification of this development. Values of perturbations in the perigee height for the satellites Vanguard I (1958 $\beta_2$) and Explorer VI (1959 $\delta$) were computed from the resultant trigonometric series. A program was developed for computing lunar and solar perturbations with the aid of the IBM 704. This program permits the inclusion of any value of the eccentricity and of the inclination, and gives the variation of the perigee height in the form of a trigonometric series with numerical coefficients. Also, the perturbations in Hansen's coordinates have been computed; this program can be included in the existing Vanguard scheme for general oblateness perturbations. The formulas used in these computations can be applied to the development of a Hansen-type numerical theory for an artificial satellite.

## DEVELOPMENT OF THE DISTURBING FUNCTION

The importance of determining the lunar and solar perturbations in the motion of an artificial satellite was indicated by Kozai's discovery that certain long-period terms in

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the development of the disturbing function cause large perturbations in the elements and that, in this way, the orbital lifetime of the satellite can be considerably affected (Reference 1). In the present treatment, the analytical development of the two main terms of the disturbing function are expressed

$$\frac{m' r^2}{r'^3}\left(\frac{3}{2} S^2 - \frac{1}{2}\right) + \frac{m' r^3}{r'^4}\left(\frac{5}{2} S^3 - \frac{3}{2} S\right),$$

where

$$S = \cos(\mathbf{r}, \mathbf{r}') .$$

In this expression $(\mathbf{r}, \mathbf{r}')$ is the angle between the position vectors of the satellite and of the perturbing body, either the sun or the moon, with respect to the earth. The main term is the second Legendre polynomial in the harmonic expansion of the disturbing function. The third Legendre polynomial is known as the parallactic term.

The analytical development is important because it permits investigation of the perturbations for an entire group of satellites having similar elements and, in particular, is useful in investigating problems connected with the effects of resonance, i.e., stability problems.

The problem of developing the disturbing effects caused by the sun and the moon is related to the lunar problem but with the difference that the orbital inclinations of both the disturbed and the disturbing body to the basic reference plane can be large in the case considered here. The arrangement given for the development of the disturbing function, based on Cayley's work (Reference 2), permits the inclusion of any power of the eccentricity and is valid for all inclinations (Reference 3).

For the time being, only long-period terms are evaluated, and the terms depending on the mean anomaly of the satellite are excluded. The resultant "abbreviated version" is given on pages 25 through 27. This last development was used to investigate the influence of the combined effect of drag and lunar and solar perturbations on the orbital lifetime of the satellite and in further investigations of the orbits of satellites in the NASA space research program. Future satellite orbits undoubtedly will have elements such that the terms depending on the mean anomaly of the satellite and the higher powers of the eccentricity can be expected to become more important, especially regarding the possible development of resonances associated with the commensurability of satellite and lunar periods.

The perturbations may also be developed by a purely numerical method based on the use of fast computing machines. In this method the values of the orbit elements $a$, $e$, $i$, $a'$, $e'$, $i'$ are substituted into the coefficients. The numerical method is convenient in developing the perturbations according to Hansen's theory and in investigating the variation of the perigee distance. A program for the development in terms of the eccentric anomaly and a program for the development in terms of the mean anomaly have been prepared. Both programs can help to supply information concerning the optimum condition for launching in connection with the solar and lunar perturbations.

Direct numerical integration of the equations of motion with the solar and lunar perturbations, by means of a computer program, has confirmed the results of the analytical treatment and the Fourier series development.

## COMPUTATION OF SATELLITE PERTURBATIONS

The following notations are used:

$a$ = semimajor axis of satellite orbit

$e$ = eccentricity

$i$ = angle of inclination to equatorial plane

$\omega$ = argument of perigee

$\Omega$ = right ascension of ascending node

$g$ = mean anomaly

$\mathbf{r}$ = position vector

$x, y, z$ = rectangular coordinates

$r$ = radius

$f$ = true anomaly

$\gamma = \sin \frac{i}{2}$ .

The corresponding elements of the disturbing body are designated by primes: $a'$, $e'$, etc. The equatorial plane is taken as the basic reference plane, and the precession and nutation of the earth's axis are neglected. The disturbing function has the form

$$\Omega = m' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{\mathbf{r}' \cdot \mathbf{r}}{r'^3} \right), \tag{1}$$

or

$$\Omega = \frac{m' r^2}{r'^3} P_2(S) + \frac{m' r^3}{r'^4} P_3(S) + \cdots . \tag{2}$$

In the latter expression

$$
\begin{aligned}
S = \cos(\mathbf{r}, \mathbf{r}') = {} & (1-\gamma^2)(1-\gamma'^2)\cos(f+\omega+\theta-f'-\omega') \\
& + \gamma^2(1-\gamma'^2)\cos(f+\omega-\theta+f'+\omega') \\
& + \gamma'^2(1-\gamma^2)\cos(f+\omega+\theta+f'+\omega') \\
& + \gamma^2\gamma'^2 \cos(f+\omega-\theta-f'-\omega') \\
& + 2\gamma\gamma'\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f+\omega-f'-\omega') \\
& - 2\gamma\gamma'\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f+\omega+f'+\omega'),
\end{aligned} \tag{3}
$$

where

$$
\theta = \Omega - \Omega',
$$

and $P_2$, $P_3$, $\cdots$ are Legendre polynomials. Only the first two terms of Equation 2 are considered:

$$
a\Omega = \frac{m'a^3}{a'^3}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3\left(\frac{3}{2}S^2 - \frac{1}{2}\right) + \frac{m'a^4}{a'^4}\left(\frac{r}{a}\right)^3\left(\frac{a'}{r'}\right)^4\left(\frac{5}{2}S^3 - \frac{3}{2}S\right). \tag{4}
$$

If $i' \neq 0$, then it is convenient to represent Equation 4 in the form of Radau (Reference 4):

$$
\begin{aligned}
a\Omega = {} & \frac{m'a^3}{a'^3}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3\left(\frac{1}{4}\,\frac{r^2-3z^2}{r^2}\,\frac{r'^2-3z'^2}{r'^2}\right. \\
& \left. + \frac{3}{4}\,\frac{x^2-y^2}{r^2}\,\frac{x'^2-y'^2}{r'^2} + 3\,\frac{xy}{r^2}\,\frac{x'y'}{r'^2} + 3\,\frac{xz}{r^2}\,\frac{x'z'}{r'^2} + 3\,\frac{yz}{r^2}\,\frac{y'z'}{r'^2}\right) \\
& + \frac{m'a^4}{a'^4}\left(\frac{r}{a}\right)^3\left(\frac{a'}{r'}\right)^4\left(\frac{3}{8}\,\frac{r^2x-5xz^2}{r^3}\,\frac{r'^2x'-5x'z'^2}{r'^3}\right. \\
& + \frac{3}{8}\,\frac{r^2y-5yz^2}{r^3}\,\frac{r'^2y'-5y'z'^2}{r'^3} + \frac{1}{4}\,\frac{3r^2z-5z^3}{r^3}\,\frac{3r'^2z'-5z'^3}{r'^3} \\
& + \frac{5}{8}\,\frac{x^3-3xy^2}{r^3}\,\frac{x'^3-3x'y'^2}{r'^3} \\
& + \frac{5}{8}\,\frac{3x^2y-y^3}{r^3}\,\frac{3x'^2y'-y'^3}{r'^3} \\
& + \frac{15}{4}\,\frac{x^2z-y^2z}{r^3}\,\frac{x'^2z'-y'^2z}{r'^3} \\
& \left. + 15\,\frac{xyz}{r^3}\,\frac{x'y'z'}{r'^3}\right);
\end{aligned} \tag{5}
$$

and to substitute

$$\frac{x}{r} = (1 - \gamma^2)\cos(f + \omega + \theta) + \gamma^2 \cos(f + \omega - \theta),$$

$$\frac{y}{r} = (1 - \gamma^2)\sin(f + \omega + \theta) - \gamma^2 \sin(f + \omega - \theta),$$

$$\frac{z}{r} = 2\gamma\sqrt{1 - \gamma^2}\,\sin(f + \omega),$$

$$\frac{x'}{r'} = \cos(f' + \omega'),$$

$$\frac{y'}{r'} = (1 - 2\gamma'^2)\sin(f' + \omega'),$$

$$\frac{z'}{r'} = 2\gamma'\sqrt{1 - \gamma'^2}\,\sin(f' + \omega').$$

Substitution of these expressions into Equation 5 results in the following development for $\Omega$ in terms of the true anomalies:

$$a\Omega = \frac{m'a^3}{a'^3}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3 \times (\textit{sum of terms of Types I-V})$$

$$+ \frac{m'a^4}{a'^4}\left(\frac{r}{a}\right)^3\left(\frac{a'}{r'}\right)^4 \times (\textit{sum of terms of Types VI-XIII}): \tag{6}$$

*Type I*

$$+\frac{1}{4}(1 - 6\gamma^2 + 6\gamma^4)(1 - 6\gamma'^2 + 6\gamma'^4)$$

$$+ 3(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)\cos 2\theta$$

$$+ 3(\gamma - 2\gamma^3)(\gamma' - 2\gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}\cos\theta;$$

*Type II*

$$+\frac{9}{2}(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)\cos(2f + 2\omega - 2f' - 2\omega')$$

$$+\frac{3}{4}(1 - \gamma^2)^2(1 - \gamma'^2)^2\cos(2f + 2\omega - 2f' - 2\omega' + 2\theta)$$

$$+\frac{3}{4}\gamma^4\gamma'^4\cos(2f + 2\omega - 2f' - 2\omega' - 2\theta)$$

$$+ 3(\gamma - \gamma^3)(\gamma' - \gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}\cos(2f + 2\omega - 2f' - 2\omega' + \theta)$$

$$+ 3\gamma^3\gamma'^3\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}\cos(2f + 2\omega - 2f' - 2\omega' - \theta);$$

*Type III*

$$+ \frac{3}{2} (\gamma^2 - \gamma^4) (1 - 6\gamma'^2 + 6\gamma'^4) \cos (2f + 2\omega)$$

$$+ \frac{3}{2} (1 - \gamma^2)^2 (\gamma'^2 - \gamma'^4) \cos (2f + 2\omega + 2\theta)$$

$$+ \frac{3}{2} \gamma^4 (\gamma'^2 - \gamma'^4) \cos (2f + 2\omega - 2\theta)$$

$$- 3 (\gamma - \gamma^3) (\gamma' - 2\gamma'^3) \sqrt{1 - \gamma^2} \sqrt{1 - \gamma'^2} \cos (2f + 2\omega + \theta)$$

$$+ 3 \gamma^3 (\gamma' - 2\gamma'^3) \sqrt{1 - \gamma^2} \sqrt{1 - \gamma'^2} \cos (2f + 2\omega - \theta) ;$$

*Type IV*

$$+ \frac{3}{2} (\gamma'^2 - \gamma'^4) (1 - 6\gamma^2 + 6\gamma^4) \cos (2f' + 2\omega')$$

$$+ \frac{3}{2} (\gamma^2 - \gamma^4) (1 - \gamma'^2)^2 \cos (2f' + 2\omega' - 2\theta)$$

$$+ \frac{3}{2} (\gamma^2 - \gamma^4) \gamma'^4 \cos (2f' + 2\omega' + 2\theta)$$

$$+ 3 (\gamma - 2\gamma^3) \gamma'^3 \sqrt{1 - \gamma^2} \sqrt{1 - \gamma'^2} \cos (2f' + 2\omega' + \theta)$$

$$- 3 (\gamma - 2\gamma^3) (\gamma' - \gamma'^3) \sqrt{1 - \gamma^2} \sqrt{1 - \gamma'^2} \cos (2f' + 2\omega' - \theta) ;$$

*Type V*

$$+ \frac{9}{2} (\gamma^2 - \gamma^4) (\gamma'^2 - \gamma'^4) \cos (2f + 2\omega + 2f' + 2\omega')$$

$$+ \frac{3}{4} (1 - \gamma^2)^2 \gamma'^4 \cos (2f + 2\omega + 2f' + 2\omega' + 2\theta)$$

$$+ \frac{3}{4} \gamma^4 (1 - \gamma'^2)^2 \cos (2f + 2\omega + 2f' + 2\omega' - 2\theta)$$

$$- 3 (\gamma - \gamma^3) \gamma'^3 \sqrt{1 - \gamma^2} \sqrt{1 - \gamma'^2} \cos (2f + 2\omega + 2f' + 2\omega' + \theta)$$

$$- 3 \gamma^3 (\gamma' - \gamma'^3) \sqrt{1 - \gamma^2} \sqrt{1 - \gamma'^2} \cos (2f + 2\omega + 2f' + 2\omega' - \theta) ;$$

*Type VI*

$$-\frac{9}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma' - 5\gamma'^3 + 5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega + f' + \omega')$$

$$+\frac{3}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(6\gamma'^2 - 20\gamma'^4 + 15\gamma'^6)\cos(f + \omega + f' + \omega' + \theta)$$

$$+\frac{3}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(1-\gamma'^2)(1 - 10\gamma'^2 + 15\gamma'^4)\cos(f + \omega + f' + \omega' - \theta)$$

$$+\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)(2\gamma'^3 - 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega + f' + \omega' + 2\theta)$$

$$+\frac{15}{4}(2\gamma^3 - 3\gamma^5)(\gamma' - 4\gamma'^3 + 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega + f' + \omega' - 2\theta)$$

$$+\frac{45}{8}(\gamma^2 - 2\gamma^4 + \gamma^6)(\gamma'^4 - \gamma'^6)\cos(f + \omega + f' + \omega' + 3\theta)$$

$$+\frac{45}{8}(\gamma^4 - \gamma^6)(\gamma'^2 - 2\gamma'^4 + \gamma'^6)\cos(f + \omega + f' + \omega' - 3\theta);$$

*Type VII*

$$+\frac{9}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma' - 5\gamma'^3 + 5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega - f' - \omega')$$

$$+\frac{3}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(1-\gamma'^2)(1 - 10\gamma'^2 + 15\gamma'^4)\cos(f + \omega - f' - \omega' + \theta)$$

$$+\frac{3}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(6\gamma'^2 - 20\gamma'^4 + 15\gamma'^6)\cos(f + \omega - f' - \omega' - \theta)$$

$$+\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)(\gamma' - 4\gamma'^3 + 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega - f' - \omega' + 2\theta)$$

$$+\frac{15}{4}(2\gamma^3 - 3\gamma^5)(2\gamma'^3 - 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega - f' - \omega' - 2\theta)$$

$$+\frac{45}{8}(\gamma^2 - 2\gamma^4 + \gamma^6)(\gamma'^2 - 2\gamma'^4 + \gamma'^6)\cos(f + \omega - f' - \omega' + 3\theta)$$

$$+\frac{45}{8}(\gamma^4 - \gamma^6)(\gamma'^4 - \gamma'^6)\cos(f + \omega - f' - \omega' - 3\theta);$$

*Type VIII*

$$-\frac{15}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma'^3 - \gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega + 3f' + 3\omega')$$

$$+\frac{15}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(\gamma'^4 - \gamma'^6)\cos(f + \omega + 3f' + 3\omega' + \theta)$$

$$+\frac{15}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(\gamma' - \gamma'^3)^2\cos(f + \omega + 3f' + 3\omega' - \theta)$$

$$+\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega + 3f' + 3\omega' + 2\theta)$$

$$-\frac{15}{4}(2\gamma^3 - 3\gamma^5)\gamma'(1-\gamma'^2)^2\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f + \omega + 3f' + 3\omega' - 2\theta)$$

$$+\frac{15}{8}(\gamma - \gamma^3)^2\gamma'^6\cos(f + \omega + 3f' + 3\omega' + 3\theta)$$

$$+\frac{15}{8}(\gamma^4 - \gamma^6)(1-\gamma'^2)^3\cos(f + \omega + 3f' + 3\omega' - 3\theta);$$

*Type IX*

$$+\frac{15}{2}(\gamma-5\gamma^3+5\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f+\omega-3f'-3\omega')$$
$$+\frac{15}{8}(1-\gamma^2)(1-10\gamma^2+15\gamma^4)(\gamma'-\gamma'^3)^2\cos(f+\omega-3f'-3\omega'+\theta)$$
$$+\frac{15}{8}(6\gamma^2-20\gamma^4+15\gamma^6)(\gamma'^4-\gamma'^6)\cos(f+\omega-3f'-3\omega'-\theta)$$
$$-\frac{15}{4}(\gamma-4\gamma^3+3\gamma^5)(\gamma'-2\gamma'^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f+\omega-3f'-3\omega'+2\theta)$$
$$+\frac{15}{4}(2\gamma^3-3\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(f+\omega-3f'-3\omega'-2\theta)$$
$$+\frac{15}{8}\gamma^2(1-\gamma^2)^2(1-\gamma'^2)^3\cos(f+\omega-3f'-3\omega'+3\theta)$$
$$+\frac{15}{8}(\gamma^4-\gamma^6)\gamma'^6\cos(f+\omega-3f'-3\omega'-3\theta);$$

*Type X*

$$-\frac{15}{2}(\gamma^3-\gamma^5)(\gamma'-5\gamma'^3+5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega+f'+\omega')$$
$$+\frac{15}{8}(\gamma-\gamma^3)^2(6\gamma'^2-20\gamma'^4+15\gamma'^6)\cos(3f+3\omega+f'+\omega'+\theta)$$
$$+\frac{15}{8}(\gamma^4-\gamma^6)(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)\cos(3f+3\omega+f'+\omega'-\theta)$$
$$-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(2\gamma'^3-3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega+f'+\omega'+2\theta)$$
$$+\frac{15}{4}\gamma^5(\gamma'-4\gamma'^3+3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega+f'+\omega'-2\theta)$$
$$+\frac{15}{8}(1-\gamma^2)^3(\gamma'^4-\gamma'^6)\cos(3f+3\omega+f'+\omega'+3\theta)$$
$$+\frac{15}{8}\gamma^6(\gamma'-\gamma'^3)^2\cos(3f+3\omega+f'+\omega'-3\theta);$$

*Type XI*

$$+\frac{15}{2}(\gamma^3-\gamma^5)(\gamma'-5\gamma'^3+5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega-f'-\omega')$$
$$+\frac{15}{8}(\gamma-\gamma^3)^2(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)\cos(3f+3\omega-f'-\omega'+\theta)$$
$$+\frac{15}{8}(\gamma^4-\gamma^6)(6\gamma'^2-20\gamma'^4+15\gamma'^6)\cos(3f+3\omega-f'-\omega'-\theta)$$
$$-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(\gamma'-4\gamma'^3+3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega-f'-\omega'+2\theta)$$
$$+\frac{15}{4}\gamma^5(2\gamma'^3-3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega-f'-\omega'-2\theta)$$
$$+\frac{15}{8}(1-\gamma^2)^3(\gamma'-\gamma'^3)^2\cos(3f+3\omega-f'-\omega'+3\theta)$$
$$+\frac{15}{8}\gamma^6(\gamma'^4-\gamma'^6)\cos(3f+3\omega-f'-\omega'-3\theta);$$

*Type XII*

$$-\frac{25}{2}(\gamma^3-\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega+3f'+3\omega')$$
$$+\frac{75}{8}(\gamma-\gamma^3)^2(\gamma'^4-\gamma'^6)\cos(3f+3\omega+3f'+3\omega'+\theta)$$
$$+\frac{75}{8}(\gamma^4-\gamma^6)(\gamma'-\gamma'^3)^2\cos(3f+3\omega+3f'+3\omega'-\theta)$$
$$-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega+3f'+3\omega'+2\theta)$$
$$-\frac{15}{4}\gamma^5(\gamma'-2\gamma'^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega+3f'+3\omega'-2\theta)$$
$$+\frac{5}{8}(1-\gamma^2)^3\gamma'^6\cos(3f+3\omega+3f'+3\omega'+3\theta)$$
$$+\frac{5}{8}\gamma^6(1-\gamma'^2)^3\cos(3f+3\omega+3f'+3\omega'-3\theta)\,;$$

*Type XIII*

$$+\frac{25}{2}(\gamma^3-\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega-3f'-3\omega')$$
$$+\frac{75}{8}(\gamma^2-2\gamma^4+\gamma^6)(\gamma'^2-2\gamma'^4+\gamma'^6)\cos(3f+3\omega-3f'-3\omega'+\theta)$$
$$+\frac{75}{8}(\gamma^4-\gamma^6)(\gamma'^4-\gamma'^6)\cos(3f+3\omega-3f'-3\omega'-\theta)$$
$$+\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(\gamma'-2\gamma^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega-3f'-3\omega'+2\theta)$$
$$+\frac{15}{4}\gamma^5\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cos(3f+3\omega-3f'-3\omega'-2\theta)$$
$$+\frac{5}{8}(1-\gamma^2)^3(1-\gamma'^2)^3\cos(3f+3\omega-3f'-3\omega'+3\theta)$$
$$+\frac{5}{8}\gamma^6\gamma'^6\cos(3f+3\omega-3f'-3\omega'-3\theta)\,.$$

Then, by using Cayley's notations, we obtain

$$\left(\frac{r}{a}\right)^p\cos(qf+\alpha)=\sum_{-\infty}^{+\infty}\left[\cos+\sin\right]^{i}_{p,q}\cos(ig+\alpha)\,,$$

$$\left(\frac{r'}{a'}\right)^{p'}\cos(q'f'+\alpha')=\sum_{-\infty}^{+\infty}\left[\cos+\sin\right]^{i'}_{p',q'}\cos(i'g'+\alpha')\,.$$

Evidently

$$\left(\frac{r}{a}\right)^p\left(\frac{r'}{a'}\right)^{p'}\cos(qf+q'f'+\alpha)$$
$$=\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}\left[\cos+\sin\right]^{i}_{p,q}\left[\cos+\sin\right]^{i'}_{p',q'}\cos(ig+i'g'+\alpha)\,. \qquad (7)$$

**(XIII)** $\left(\frac{r}{a}\right)^3\left(\frac{a'}{r'}\right)^4\cos(3f-3f'+\alpha)\,,$

The coefficients $\left[\cos + \sin\right]^{i}_{p,q}$ and $\left[\cos + \sin\right]^{i'}_{p',q'}$ are functions of the eccentricities and can be taken, in a general case, from Cayley's tables (Reference 5).

For the development of

$$\left(\frac{r}{a}\right)^{p}\left(\frac{r'}{a'}\right)^{p'} \cos\,(qf + q'f' + \imath)\ ,$$



*On the top, the factors depending on $\gamma$ and $\gamma'$ are given. Each factor is a product of a polynomial in $\gamma$ and a polynomial in $\gamma'$. These factors are analogous to the polynomials of Tisserand from the planetary theories. On the right are the angles that correspond to the $\gamma$, $\gamma'$ factors and that must be added to the arguments. Each coefficient above must be multiplied by each coefficient below, and the arguments are added. The common factors, $m'a^3/a'^3$ for Types I-V and $m'a^4/a'^4$ for Types VI-XIII, are omitted.*

As an example, from the arrangement for Type $a\Omega_1$ (below) we take, to order $e^2$ and $e'^2$,

$$3\,(\gamma - 2\gamma^3)\,(\gamma' - 2\gamma'^3)\,\sqrt{1-\gamma^2}\,\sqrt{1-\gamma'^2}\,\frac{m'a^3}{a'^3}\quad\frac{9}{4}e'^2\left(-\frac{1}{4}e^2\right)\cos\,(+\,2g + 2g' + \theta)\ .$$

There is no necessity to perform the actual multiplication; it can be done separately in each case.

*Type* $a\Omega_1$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{1}{4}(1 - 6\gamma^2 + 6\gamma^4)(1 - 6\gamma'^2 + 6\gamma'^4)$ | $2\tau = 0$ |
| $+\,3\,(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)$ | $2\tau = 2\theta$ |
| $+\,3\,(\gamma - 2\gamma^3)(\gamma' - 2\gamma'^3)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $2\tau = \theta$ |

| | | |
|---|---|---|
| | ..................... | .......... |
| | $+\frac{9}{4}e'^2 + \frac{7}{4}e'^4 + \cdots$ | $-\,2g'$ |
| | $+\frac{3}{2}e' + \frac{27}{16}e'^3 + \cdots$ | $-\,g'$ |
| | $(1 - e'^2)^{-3/2}$ | $0.g'$ |
| | $+\frac{3}{2}e' + \frac{27}{16}e'^3 + \cdots$ | $+\,g'$ |
| | $+\frac{9}{4}e'^2 + \frac{7}{4}e'^4 + \cdots$ | $+\,2g'$ |
| | ..................... | .......... |
| ..................... | | .......... |
| $-\frac{1}{4}e^2 + \frac{1}{12}e^4 + \cdots$ | | $-\,2g + 2\tau$ |

## *Type* $a\Omega_2$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma^2-\gamma^4)(\gamma'^2-\gamma'^4)$ | $2\tau = 2\omega$ |
| $+\frac{3}{4}(1-\gamma^2)^2(1-\gamma'^2)^2$ | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{4}\gamma^4\gamma'^4$ | $2\tau = 2\omega - 2\theta$ |
| $+3(\gamma-\gamma^3)(\gamma'-\gamma'^3)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $2\tau = 2\omega + \theta$ |
| $+3\gamma^3\gamma'^3\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $2\tau = 2\omega - \theta$ |

| | | |
|---|---|---|
| | ........................ | ........................ |
| | $+\frac{17}{2}e'^2-\frac{115}{6}e'^4+\cdots$ | $-4g'-2\omega'$ |
| | $+\frac{7}{2}e'-\frac{123}{16}e'^3+\cdots$ | $-3g'-2\omega'$ |
| | $1-\frac{5}{2}e'^2+\frac{13}{16}e'^4+\cdots$ | $-2g'-2\omega'$ |
| | $-\frac{1}{2}e'+\frac{1}{16}e'^3+\cdots$ | $-g'-2\omega'$ |
| | 0 *(exact)* | $0.g'-2\omega'$ |
| | $+\frac{1}{48}e'^3+\cdots$ | $+g'-2\omega'$ |
| | $+\frac{1}{24}e'^4+\cdots$ | $+2g'-2\omega'$ |
| | ........................ | ........................ |
| ........................ | | ........................ |
| $-\frac{1}{16}e^4-\frac{11}{480}e^6$ | | $-2g+2\tau$ |
| $-\frac{7}{24}e^3-\frac{47}{384}e^5+\cdots$ | | $-g+2\tau$ |
| $+\frac{5}{2}e^2$ *(exact)* | | $0.g+2\tau$ |
| $-3e+\frac{13}{8}e^3+\cdots$ | | $+g+2\tau$ |
| $1-\frac{5}{2}e^2+\frac{23}{16}e^4+\cdots$ | | $+2g+2\tau$ |
| ........................ | | ........................ |

## *Type* $a\Omega_3$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{3}{2}(\gamma^2-\gamma^4)(1-6\gamma'^2+6'^4)$ | $2\tau = 2\omega$ |
| $+\frac{3}{2}(1-\gamma^2)^2(\gamma'^2-\gamma'^4)$ | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{2}\gamma^4(\gamma'^2-\gamma'^4)$ | $2\tau = 2\omega - 2\theta$ |
| $+3\gamma^3(\gamma'-2\gamma'^3)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $2\tau = 2\omega - \theta$ |
| $-3(\gamma-\gamma^3)(\gamma'-2\gamma'^3)\sqrt{1-\gamma^{\,2}}\sqrt{1-\gamma'^2}$ | $2\tau = 2\omega + \theta$ |

| | | |
|---|---|---|
| | ........................ | ...................... |
| | $+\frac{9}{4}e'^2+\frac{7}{4}e'^4+\cdots$ | $-2g'$ |
| | $+\frac{3}{2}e'+\frac{27}{16}e'^3+\cdots$ | $-g'$ |
| | $1+\frac{3}{2}e'^2+\frac{15}{8}e'^4+\cdots$ | $0.g'$ |
| | $+\frac{3}{2}e'+\frac{27}{16}e'^3+\cdots$ | $+g'$ |
| | $+\frac{9}{4}e'^2+\frac{7}{4}e'^4+\cdots$ | $+2g'$ |
| | ........................ | ...................... |
| ...................... | | ...................... |
| $-\frac{1}{16}e^4+\cdots$ | | $-2g+2\tau$ |
| $-\frac{7}{24}e^3+\cdots$ | | $-g+2\tau$ |
| $+\frac{5}{2}e^2$ *(exact)* | | $0.g+2\tau$ |
| $-3e+\frac{13}{8}e^3+\cdots$ | | $+g+2\tau$ |
| $1-\frac{5}{2}e^2+\frac{23}{16}e^4+\cdots$ | | $+2g+2\tau$ |
| ...................... | | ...................... |

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## *Type* $a\Omega_4$

| ***Factors*** | ***cos terms*** |
|---|---|
| $+\frac{3}{2}(1 - 6\gamma^2 + 6\gamma^4)(\gamma'^2 - \gamma'^4)$ | $2\tau' = 2\omega'$ |
| $+\frac{3}{2}(\gamma^2 - \gamma^4)\gamma'^4$ | $2\tau' = 2\omega' + 2\theta$ |
| $+\frac{3}{2}(\gamma^2 - \gamma^4)(1 - \gamma'^2)^2$ | $2\tau' = 2\omega' - 2\theta$ |
| $+3(\gamma - 2\gamma^3)\gamma'^3\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $2\tau' = 2\omega' + \theta$ |
| $-3(\gamma - 2\gamma^3)(\gamma' - \gamma'^3)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $2\tau' = 2\omega' - \theta$ |

| | | |
|---|---|---|
| | ........................ | ..................... |
| | $+\frac{1}{48}e'^3 + \cdots$ | $-g' + 2\tau'$ |
| | $0$ *(exact)* | $0.g' + 2\tau'$ |
| | $-\frac{1}{2}e' + \frac{1}{16}e'^3 + \cdots$ | $+g' + 2\tau'$ |
| | $1 - \frac{5}{2}e'^2 + \frac{13}{16}e'^4 + \cdots$ | $+2g' + 2\tau'$ |
| | $+\frac{7}{2}e' - \frac{123}{16}e'^3 + \cdots$ | $+3g' + 2\tau'$ |
| | $+\frac{17}{2}e'^2 - \frac{115}{16}e'^4 + \cdots$ | $+4g' + 2\tau'$ |
| | ........................ | ..................... |
| ..................... | | ..................... |
| $-\frac{1}{4}e^2 + \frac{1}{12}e^4 + \cdots$ | | $-2g$ |
| $-e + \frac{1}{8}e^3 + \cdots$ | | $-g$ |
| $1 + \frac{3}{2}e^2$ *(exact)* | | $0.g$ |
| $-e + \frac{1}{8}e^3 + \cdots$ | | $+g$ |
| $-\frac{1}{4}e^2 + \frac{1}{12}e^4 + \cdots$ | | $+2g$ |
| ..................... | | ..................... |

## *Type* $a\Omega_5$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)$ | $2\tau = 2\omega$ |
| $+\frac{3}{4}(1 - \gamma^2)^2 \gamma'^4$ | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{4}\gamma^4(1 - \gamma'^2)^2$ | $2\tau = 2\omega - 2\theta$ |
| $- 3(\gamma - \gamma^3)\ \gamma'^3 \sqrt{1-\gamma^2}\ \sqrt{1-\gamma'^2}$ | $2\tau = 2\omega + \theta$ |
| $- 3\gamma^3(\gamma' - \gamma'^3) \sqrt{1-\gamma^2}\ \sqrt{1-\gamma'^2}$ | $2\tau = 2\omega - \theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $+\frac{1}{48}e'^3 + \cdots$ | $- g' + 2\omega'$ |
| | $0$ *(exact)* | $0.g' + 2\omega'$ |
| | $-\frac{1}{2}e' + \frac{1}{16}e'^3 + \cdots$ | $+ g' + 2\omega'$ |
| | $1 - \frac{5}{2}e'^2 + \frac{13}{16}e'^4 + \cdots$ | $+ 2g' + 2\omega'$ |
| | $+\frac{7}{2}e' - \frac{123}{16}e'^3 + \cdots$ | $+ 3g' + 2\omega'$ |
| | $+\frac{17}{2}e'^2 - \frac{115}{6}e'^4 + \cdots$ | $+ 4g' + 2\omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $-\frac{7}{24}e + \cdots$ | | $- g + 2\tau$ |
| $+\frac{5}{2}e^2$ *(exact)* | | $0.g + 2\tau$ |
| $- 3e + \frac{13}{8}e^3 + \cdots$ | | $+ g + 2\tau$ |
| $1 - \frac{5}{2}e^2 + \frac{23}{16}e^4 + \cdots$ | | $+ 2g + 2\tau$ |
| $+ e - \frac{19}{8}e^3 + \cdots$ | | $+ 3g + 2\tau$ |
| $+ e^2 - \frac{5}{2}e^4 + \cdots$ | | $+ 4g + 2\tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_6$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{9}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma' - 5\gamma'^3 + 5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega$ |
| $+\frac{3}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(6\gamma'^2 - 20\gamma'^4 + 15\gamma'^6)$ | $\tau = \omega + \theta$ |
| $+\frac{3}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(1-\gamma'^2)(1 - 10\gamma'^2 + 15\gamma'^4)$ | $\tau = \omega - \theta$ |
| $+\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)(2\gamma'^3 - 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega + 2\theta$ |
| $+\frac{15}{4}(2\gamma^3 - 3\gamma^5)(\gamma' - 4\gamma'^3 + 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega - 2\theta$ |
| $+\frac{45}{8}(\gamma^2 - 2\gamma^4 + \gamma^6)(\gamma'^4 - \gamma'^6)$ | $\tau = \omega + 3\theta$ |
| $+\frac{45}{8}(\gamma^4 - \gamma^6)(\gamma'^2 - 2\gamma'^4 + \gamma'^6)$ | $\tau = \omega - 3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $+\frac{11}{8}e'^2 + \cdots$ | $-g' + \omega'$ |
| | $+e' + \cdots$ | $0.g' + \omega'$ |
| | $+1 + 2e'^2 + \cdots$ | $+g' + \omega'$ |
| | $+3e' + \cdots$ | $+2g' + \omega'$ |
| | $+\frac{53}{8}e'^2 + \cdots$ | $+3g' + \omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $+\frac{11}{8}e^2 + \cdots$ | | $-g + \tau$ |
| $-\frac{5}{2}e - \frac{15}{8}e^3$ *(exact)* | | $0.g + \tau$ |
| $+1 + 2e^2 + \cdots$ | | $+g + \tau$ |
| $-\frac{1}{2}e + \cdots$ | | $+2g + \tau$ |
| $-\frac{3}{8}e^2 + \cdots$ | | $+3g + \tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_7$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma' - 5\gamma'^3 + 5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega$ |
| $+\frac{3}{8}(1-\gamma^2)(1-10\gamma^2+15\gamma^4)(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)$ | $\tau = \omega + \theta$ |
| $+\frac{3}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(6\gamma'^2 - 20\gamma'^4 + 15\gamma'^6)$ | $\tau = \omega - \theta$ |
| $+\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)(\gamma' - 4\gamma'^3 + 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega + 2\theta$ |
| $+\frac{15}{4}(2\gamma^3 - 3\gamma^5)(2\gamma'^3 - 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega - 2\theta$ |
| $+\frac{45}{8}(\gamma^2 - 2\gamma^4 + \gamma^6)(\gamma'^2 - 2\gamma'^4 + \gamma'^6)$ | $\tau = \omega + 3\theta$ |
| $+\frac{45}{8}(\gamma^4 - \gamma^6)(\gamma'^4 - \gamma'^6)$ | $\tau = \omega - 3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $+\frac{53}{8}e'^2 + \cdots$ | $-3g' - \omega'$ |
| | $+3e' + \cdots$ | $-2g' - \omega'$ |
| | $+1 + 2e'^2 + \cdots$ | $-g' - \omega'$ |
| | $+e' + \cdots$ | $0.g' - \omega'$ |
| | $+\frac{11}{8}e'^2 + \cdots$ | $+g' - \omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $+\frac{11}{8}e^2 + \cdots$ | | $-g + \tau$ |
| $-\frac{5}{2}e - \frac{15}{8}e^3$ *(exact)* | | $0.g + \tau$ |
| $+1 + 2e^2 + \cdots$ | | $+g + \tau$ |
| $-\frac{1}{2}e + \cdots$ | | $+2g + \tau$ |
| $-\frac{3}{8}e^2 + \cdots$ | | $+3g + \tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_8$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{15}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma'^3 - \gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega$ |
| $+\frac{15}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(\gamma'^4 - \gamma'^6)$ | $\tau = \omega + \theta$ |
| $+\frac{15}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(\gamma' - \gamma'^3)^2$ | $\tau = \omega - \theta$ |
| $+\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega + 2\theta$ |
| $-\frac{15}{4}(2\gamma^3 - 3\gamma^5)\gamma'(1-\gamma'^2)^2\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega - 2\theta$ |
| $+\frac{15}{8}(\gamma - \gamma^3)^2\gamma'^6$ | $\tau = \omega + 3\theta$ |
| $+\frac{15}{8}(\gamma^4 - \gamma^6)(1-\gamma'^2)^3$ | $\tau = \omega - 3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $0$ *(exact)* | $0.g' + 3\omega'$ |
| | $+\frac{1}{8}e'^2 + \cdots$ | $+g' + 3\omega'$ |
| | $-e' + \cdots$ | $+2g' + 3\omega'$ |
| | $+1 - 6e'^2 + \cdots$ | $+3g' + 3\omega'$ |
| | $+5e' + \cdots$ | $+4g' + 3\omega'$ |
| | $+\frac{127}{8}e'^2 + \cdots$ | $+5g' + 3\omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $+\frac{11}{8}e^2 + \cdots$ | | $-g + \tau$ |
| $-\frac{5}{2}e - \frac{15}{8}e^3$ *(exact)* | | $0.g + \tau$ |
| $+1 + 2e^2 + \cdots$ | | $+g + \tau$ |
| $-\frac{1}{2}e + \cdots$ | | $+2g + \tau$ |
| $-\frac{3}{8}e^2 + \cdots$ | | $+3g + \tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_9$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{15}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma'^3 - \gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega$ |
| $+\frac{15}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(\gamma' - \gamma'^3)^2$ | $\tau = \omega + \theta$ |
| $+\frac{15}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(\gamma'^4 - \gamma'^6)$ | $\tau = \omega - \theta$ |
| $-\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)(\gamma' - 2\gamma'^3 + \gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega + 2\theta$ |
| $+\frac{15}{4}(2\gamma^3 - 3\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = \omega - 2\theta$ |
| $+\frac{15}{8}\gamma^2(1-\gamma^2)^2(1-\gamma'^2)^3$ | $\tau = \omega + 3\theta$ |
| $+\frac{15}{8}(\gamma^4 - \gamma^6)\gamma'^6$ | $\tau = \omega - 3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $+\frac{127}{8}e'^2 + \cdots$ | $-5g' - 3\omega'$ |
| | $+5e' + \cdots$ | $-4g' - 3\omega'$ |
| | $+1 - 6e'^2 + \cdots$ | $-3g' - 3\omega'$ |
| | $-e' + \cdots$ | $-2g' - 3\omega'$ |
| | $+\frac{1}{8}e'^2 + \cdots$ | $-g' - 3\omega'$ |
| | 0 *(exact)* | $0.g' - 3\omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $+\frac{11}{8}e^2 + \cdots$ | | $-g + \tau$ |
| $-\frac{5}{2}e - \frac{15}{8}e^3$ *(exact)* | | $0.g + \tau$ |
| $+1 + 2e^2 + \cdots$ | | $+g + \tau$ |
| $-\frac{1}{2}e + \cdots$ | | $+2g + \tau$ |
| $-\frac{3}{8}e^2 + \cdots$ | | $+3g + \tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_{10}$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{15}{2}(\gamma^3-\gamma^5)(\gamma'-5\gamma'^3+5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau=3\omega$ |
| $+\frac{15}{8}(\gamma-\gamma^3)^2(6\gamma'^2-20\gamma'^4+15\gamma'^6)$ | $\tau=3\omega+\theta$ |
| $+\frac{15}{8}(\gamma^4-\gamma^6)(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)$ | $\tau=3\omega-\theta$ |
| $-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(2\gamma'^3-3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau=3\omega+2\theta$ |
| $+\frac{15}{4}\gamma^5(\gamma'-4\gamma'^3+3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau=3\omega-2\theta$ |
| $+\frac{15}{8}(1-\gamma^2)^3(\gamma'^4-\gamma'^6)$ | $\tau=3\omega+3\theta$ |
| $+\frac{15}{8}\gamma^6(\gamma'-\gamma'^3)^2$ | $\tau=3\omega-3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $+\frac{11}{8}e'^2+\cdots$ | $-g'+\omega'$ |
| | $+e'+\cdots$ | $0.g'+\omega'$ |
| | $+1+2e'^2+\cdots$ | $+g'+\omega'$ |
| | $+3e'+\cdots$ | $+2g'+\omega'$ |
| | $+\frac{53}{8}e'^2+\cdots$ | $+3g'+\omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $-\frac{35}{8}e^3$ *(exact)* | | $0.g+\tau$ |
| $+\frac{57}{8}e^2+\cdots$ | | $+g+\tau$ |
| $-\frac{9}{2}e+\cdots$ | | $+2g+\tau$ |
| $+1-6e^2+\cdots$ | | $+3g+\tau$ |
| $+\frac{3}{2}e+\cdots$ | | $+4g+\tau$ |
| $+\frac{15}{8}e^2+\cdots$ | | $+5g+\tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_{11}$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{15}{2}(\gamma^3-\gamma^5)(\gamma'-5\gamma'^3+5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega$ |
| $+\frac{15}{8}(\gamma-\gamma^3)^2(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)$ | $\tau = 3\omega+\theta$ |
| $+\frac{15}{8}(\gamma^4-\gamma^6)(6\gamma'^2-20\gamma'^4+15\gamma'^6)$ | $\tau = 3\omega-\theta$ |
| $-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(\gamma'-4\gamma'^3+3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega+2\theta$ |
| $+\frac{15}{4}\gamma^5(2\gamma'^3-3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega-2\theta$ |
| $+\frac{15}{8}(1-\gamma^2)^3(\gamma'-\gamma'^3)^2$ | $\tau = 3\omega+3\theta$ |
| $+\frac{15}{8}\gamma^6(\gamma'^4-\gamma'^6)$ | $\tau = 3\omega-3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | $+\frac{53}{8}e'^2+\cdots$ | $-3g'-\omega'$ |
| | $+3e'+\cdots$ | $-2g'-\omega'$ |
| | $+1+2e'^2+\cdots$ | $-g'-\omega'$ |
| | $+e'+\cdots$ | $0.g'-\omega'$ |
| | $+\frac{11}{8}e'^2+\cdots$ | $+g'-\omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $-\frac{35}{8}e^3$ *(exact)* | | $0.g+\tau$ |
| $+\frac{57}{8}e^2+\cdots$ | | $+g+\tau$ |
| $-\frac{9}{2}e+\cdots$ | | $+2g+\tau$ |
| $+1-6e^2+\cdots$ | | $+3g+\tau$ |
| $+\frac{3}{2}e+\cdots$ | | $+4g+\tau$ |
| $+\frac{15}{8}e^2+\cdots$ | | $+5g+\tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_{12}$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{25}{2}(\gamma^3-\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega$ |
| $+\frac{75}{8}(\gamma-\gamma^3)^2(\gamma'^4-\gamma'^6)$ | $\tau = 3\omega + \theta$ |
| $+\frac{75}{8}(\gamma^4-\gamma^6)(\gamma'-\gamma'^3)^2$ | $\tau = 3\omega - \theta$ |
| $-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega + 2\theta$ |
| $-\frac{15}{4}\gamma^5(\gamma'-2\gamma'^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega - 2\theta$ |
| $+\frac{5}{8}(1-\gamma^2)^3\gamma'^6$ | $\tau = 3\omega + 3\theta$ |
| $+\frac{5}{8}\gamma^6(1-\gamma'^2)^3$ | $\tau = 3\omega - 3\theta$ |

| | | |
|---|---|---|
| | ...................... | ...................... |
| | 0 *(exact)* | $0.g' + 3\omega'$ |
| | $+\frac{1}{8}e'^2 + \cdots$ | $+ g' + 3\omega'$ |
| | $-e' + \cdots$ | $+ 2g' + 3\omega'$ |
| | $+1 - 6e'^2 + \cdots$ | $+ 3g' + 3\omega'$ |
| | $+5e' + \cdots$ | $+ 4g' + 3\omega'$ |
| | $+\frac{127}{8}e'^2 + \cdots$ | $+ 5g' + 3\omega'$ |
| | ...................... | ...................... |
| ...................... | | ...................... |
| $-\frac{35}{8}e^3$ *(exact)* | | $0.g + \tau$ |
| $+\frac{57}{8}e^2 + \cdots$ | | $+ g + \tau$ |
| $-\frac{9}{2}e + \cdots$ | | $+ 2g + \tau$ |
| $+1 - 6e^2 + \cdots$ | | $+ 3g + \tau$ |
| $+\frac{3}{2}e + \cdots$ | | $+ 4g + \tau$ |
| $+\frac{15}{8}e^2 + \cdots$ | | $+ 5g + \tau$ |
| ...................... | | ...................... |

## *Type* $a\Omega_{13}$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{25}{2}(\gamma^3-\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega$ |
| $+\frac{75}{8}(\gamma^2-2\gamma^4+\gamma^6)(\gamma'^2-2\gamma'^4+\gamma'^6)$ | $\tau = 3\omega+\theta$ |
| $+\frac{75}{8}(\gamma^4-\gamma^6)(\gamma'^4-\gamma'^6)$ | $\tau = 3\omega-\theta$ |
| $+\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(\gamma'-2\gamma'^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega+2\theta$ |
| $+\frac{15}{4}\gamma^5\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$ | $\tau = 3\omega-2\theta$ |
| $+\frac{5}{8}(1-\gamma^2)^3(1-\gamma'^2)^3$ | $\tau = 3\omega+3\theta$ |
| $+\frac{5}{8}\gamma^6\gamma'^6$ | $\tau = 3\omega-3\theta$ |

| | | |
|---|---|---|
| | .................... | .................... |
| | $+\frac{127}{8}e'^2+\cdots$ | $-5g'-3\omega'$ |
| | $+5e'+\cdots$ | $-4g'-3\omega'$ |
| | $+1-6e'^2+\cdots$ | $-3g'-3\omega'$ |
| | $-e'+\cdots$ | $-2g'-3\omega'$ |
| | $+\frac{1}{8}e'^2+\cdots$ | $-g'-3\omega'$ |
| | $0$ *(exact)* | $0.g'-3\omega'$ |
| | .................... | .................... |
| .................... | | .................... |
| $-\frac{35}{8}e^3$ *(exact)* | | $0.g+\tau$ |
| $+\frac{57}{8}e^2+\cdots$ | | $+g+\tau$ |
| $-\frac{9}{2}e+\cdots$ | | $+2g+\tau$ |
| $+1-6e^2+\cdots$ | | $+3g+\tau$ |
| $+\frac{3}{2}e+\cdots$ | | $+4g+\tau$ |
| $+\frac{15}{8}e^2+\cdots$ | | $+5g+\tau$ |
| .................... | | .................... |

However, in the case of the existing artificial satellites only the long-period terms are kept. The first power of $e'$ is kept only in the second Legendre polynomial, because of the presence of the additional power of $a/a'$ in the parallactic term. The accuracy of the coefficients of $\cos(0.g + 2\tau)$ makes the resulting development of the disturbing function valid for all eccentricities and all inclinations, provided the ratio $a/a'$ is small enough to secure fast convergence of the development of Equation 2. Thus we have the following:

*Type* $a[\Omega]_1$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{1}{4}(1 - 6\gamma^2 + 6\gamma^4)(1 - 6\gamma'^2 + 6\gamma'^4)\left(1 + \frac{3}{2}e^2\right)$, | $2\tau = 0$ |
| $+ 3(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)\left(1 + \frac{3}{2}e^2\right)$, | $2\tau = 2\theta$ |
| $+ 3(\gamma - 2\gamma^3)(\gamma' - 2\gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}\left(1 + \frac{3}{2}e^2\right)$, | $2\tau = \theta$ |

*Multiplied by*

$$+\frac{3}{2}e' \cos(2\tau - g') + (1 - e'^2)^{-\frac{3}{2}} \cos 2\tau^{(*)} + \frac{3}{2}e' \cos(2\tau + g').$$

*Type* $a[\Omega]_2$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)\cdot\frac{5}{2}e^2$, | $2\tau = 2\omega$ |
| $+\frac{3}{4}(1 - \gamma^2)^2(1 - \gamma'^2)^2\cdot\frac{5}{2}e^2$, | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{4}\gamma^4\gamma'^4\cdot\frac{5}{2}e^2$, | $2\tau = 2\omega - 2\theta$ |
| $+ 3(\gamma - \gamma^3)(\gamma' - \gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}\cdot\frac{5}{2}e^2$, | $2\tau = 2\omega + \theta$ |
| $+ 3\gamma^3\gamma'^3\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}\cdot\frac{5}{2}e^2$, | $2\tau = 2\omega - \theta$ |

*Multiplied by*

$$+\frac{7}{2}e' \cos(2\tau - 3g' - 2\omega') + \cos(2\tau - 2g' - 2\omega')$$

$$-\frac{1}{2}e' \cos(2\tau - g' - 2\omega').$$

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*The second term becomes a secular one for $2\tau = 0$.

## *Type* $a[\Omega]_3$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{3}{2}(\gamma^2 - \gamma^4)\ (1 - 6\gamma'^2 + 6\gamma'^4)\ \cdot \frac{5}{2} e^2$ , | $2\tau = 2\omega$ |
| $+\frac{3}{2}(1 - \gamma^2)^2\ (\gamma'^2 - \gamma'^4)\ \cdot \frac{5}{2} e^2$ , | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{2}\gamma^4\ (\gamma'^2 - \gamma'^4)\ \cdot \frac{5}{2} e^2$ , | $2\tau = 2\omega - 2\theta$ |
| $- 3(\gamma - \gamma^3)(\gamma' - 2\gamma'^3)\ \sqrt{1 - \gamma^2}\ \sqrt{1 - \gamma'^2}\ \cdot \frac{5}{2} e^2$ , | $2\tau = 2\omega + \theta$ |
| $+ 3\gamma^3\ (\gamma' - 2\gamma'^3)\ \sqrt{1 - \gamma^2}\ \sqrt{1 - \gamma'^2}\ \cdot \frac{5}{2} e^2$ , | $2\tau = 2\omega - \theta$ |

*Multiplied by*

$$+ \frac{3}{2} e' \cos(2\tau - g') + \cos 2\tau + \frac{3}{2} e' \cos(2\tau + g') .$$

## *Type* $a[\Omega]_4$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{3}{2}(1 - 6\gamma^2 + 6\gamma^4)\ (\gamma'^2 - \gamma'^4)\ \cdot \left(1 + \frac{3}{2} e^2\right)$ , | $2\tau = 2\omega'$ |
| $+\frac{3}{2}(\gamma^2 - \gamma^4)\ \gamma'^4\ \cdot \left(1 + \frac{3}{2} e^2\right)$ , | $2\tau = 2\omega' + 2\theta$ |
| $+\frac{3}{2}(\gamma^2 - \gamma^4)\ (1 - \gamma'^2)^2\ \cdot \left(1 + \frac{3}{2} e^2\right)$ , | $2\tau = 2\omega' - 2\theta$ |
| $+ 3(\gamma - 2\gamma^3)\ \gamma'^3\ \sqrt{1 - \gamma^2}\ \sqrt{1 - \gamma'^2}\ \cdot \left(1 + \frac{3}{2} e^2\right)$ , | $2\tau = 2\omega' + \theta$ |
| $- 3(\gamma - 2\gamma^3)\ (\gamma' - \gamma'^3)\ \sqrt{1 - \gamma^2}\ \cdot \sqrt{1 - \gamma'^2}\ \cdot \left(1 + \frac{3}{2} e^2\right)$ , | $2\tau = 2\omega' - \theta$ |

*Multiplied by*

$$- \frac{1}{2} e' \cos(2\tau + g') + \cos(2\tau + 2g') + \frac{7}{2} e' \cos(2\tau + 3g') .$$

## *Type* $a\,[\Omega]_5$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma^2-\gamma^4)\,(\gamma'^2-\gamma'^4)\cdot\frac{5}{2}e^2,$ | $2\tau = 2\omega + 2\omega'$ |
| $+\frac{3}{4}(1-\gamma^2)^2\,\gamma'^4\cdot\frac{5}{2}e^2,$ | $2\tau = 2\omega + 2\theta + 2\omega'$ |
| $+\frac{3}{4}\gamma^4\,(1-\gamma'^2)^2\cdot\frac{5}{2}e^2,$ | $2\tau = 2\omega - 2\theta + 2\omega'$ |
| $-3\,(\gamma-\gamma^3)\,\gamma'^3\,\sqrt{1-\gamma^2}\cdot\sqrt{1-\gamma'^2}\cdot\frac{5}{2}e^2,$ | $2\tau = 2\omega + \theta + 2\omega'$ |
| $-3\gamma^3\,(\gamma'-\gamma'^3)\,\sqrt{1-\gamma^2}\cdot\sqrt{1-\gamma'^2}\cdot\frac{5}{2}e^2,$ | $2\tau = 2\omega - \theta + 2\omega'$ |

*Multiplied by*

$$-\frac{1}{2}e'\cos(2\tau+g') + \cos(2\tau+2g') + \frac{7}{2}e'\cos(2\tau+3g').$$

## *Type* $a\,[\Omega]_6$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{9}{2}(\gamma-5\gamma^3+5\gamma^5)\,(\gamma'-5\gamma'^3+5\gamma'^5)\,\sqrt{1-\gamma^2}\,\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega$ |
| $+\frac{3}{8}(1-\gamma^2)\,(1-10\gamma^2+15\gamma^4)\,(6\gamma'^2-20\gamma'^4+15\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega + \theta$ |
| $+\frac{3}{8}(6\gamma^2-20\gamma^4+15\gamma^6)\,(1-\gamma'^2)\,(1-10\gamma'^2+15\gamma'^4)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega - \theta$ |
| $+\frac{15}{4}(\gamma-4\gamma^3+3\gamma^5)\,(2\gamma'^3-3\gamma'^5)\,\sqrt{1-\gamma^2}\,\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega + 2\theta$ |
| $+\frac{15}{4}(2\gamma^3-3\gamma^5)\,(\gamma'-4\gamma'^3+3\gamma'^5)\,\sqrt{1-\gamma^2}\,\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega - 2\theta$ |
| $+\frac{45}{8}(\gamma^2-2\gamma^4+\gamma^6)\,(\gamma'^4-\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega + 3\theta$ |
| $+\frac{45}{8}(\gamma^4-\gamma^6)\,(\gamma'^2-2\gamma'^4+\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau = \omega - 3\theta$ |

*Multiplied by*

$$+\cos(\tau+g'+\omega').$$

## *Type* $a[\Omega]_7$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma-5\gamma^3+5\gamma^5)(\gamma'-5\gamma'^3+5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega$ |
| $+\frac{3}{8}(1-\gamma^2)(1-10\gamma^2+15\gamma^4)(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega+\theta$ |
| $+\frac{3}{8}(6\gamma^2-20\gamma^4+15\gamma^6)(6\gamma'^2-20\gamma'^4+15\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega-\theta$ |
| $+\frac{15}{4}(\gamma-4\gamma^3+3\gamma^5)(\gamma'-4\gamma'^3+3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega+2\theta$ |
| $+\frac{15}{4}(2\gamma^3-3\gamma^5)(2\gamma'^3-3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega-2\theta$ |
| $+\frac{45}{8}(\gamma^2-2\gamma^4+\gamma^6)(\gamma'^2-2\gamma'^4+\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega+3\theta$ |
| $+\frac{45}{8}(\gamma^4-\gamma^6)(\gamma'^4-\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega-3\theta$ |

*Multiplied by*

$+\cos(\tau-g'-\omega').$

## *Type* $a[\Omega]_8$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{15}{2}(\gamma-5\gamma^3+5\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega$ |
| $+\frac{15}{8}(1-\gamma^2)(1-10\gamma^2+15\gamma^4)(\gamma'^4-\gamma'^6)\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega+\theta$ |
| $+\frac{15}{8}(6\gamma^2-20\gamma^4+15\gamma^6)(\gamma'-\gamma'^3)^2\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega-\theta$ |
| $+\frac{15}{4}(\gamma-4\gamma^3+3\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega+2\theta$ |
| $-\frac{15}{4}(2\gamma^3-3\gamma^5)\gamma'(1-\gamma'^2)^2\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega-2\theta$ |
| $+\frac{15}{8}(\gamma-\gamma^3)^2\gamma'^6\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega+3\theta$ |
| $+\frac{15}{8}(\gamma^4-\gamma^6)(1-\gamma'^2)^3\cdot\left(-\frac{5}{2}e-\frac{15}{8}e^3\right),$ | $\tau=\omega-3\theta$ |

*Multiplied by*

$+\cos(\tau+3g'+3\omega').$

### *Type* $a[\Omega]_9$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{15}{2}(\gamma - 5\gamma^3 + 5\gamma^5)(\gamma'^3 - \gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e - \frac{15}{8}e^3\right),$ | $\tau = \omega$ |
| $+\frac{15}{8}(1-\gamma^2)(1 - 10\gamma^2 + 15\gamma^4)(\gamma' - \gamma'^3)^2\cdot\left(-\frac{5}{2}e - \frac{15}{8}e^3\right),$ | $\tau = \omega + \theta$ |
| $+\frac{15}{8}(6\gamma^2 - 20\gamma^4 + 15\gamma^6)(\gamma'^4 - \gamma'^6)\cdot\left(-\frac{5}{2}e - \frac{15}{8}e^3\right),$ | $\tau = \omega - \theta$ |
| $-\frac{15}{4}(\gamma - 4\gamma^3 + 3\gamma^5)(\gamma' - 2\gamma'^3 + \gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e - \frac{15}{8}e\right),$ | $\tau = \omega + 2\theta$ |
| $+\frac{15}{4}(2\gamma^3 - 3\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{5}{2}e - \frac{15}{8}e^3\right),$ | $\tau = \omega - 2\theta$ |
| $+\frac{15}{8}\gamma^2(1-\gamma^2)^2(1-\gamma'^2)^3\cdot\left(-\frac{5}{2}e - \frac{15}{8}e^3\right),$ | $\tau = \omega + 3\theta$ |
| $+\frac{15}{8}(\gamma^4 - \gamma^6)\gamma'^6\cdot\left(-\frac{5}{2}e - \frac{15}{8}e^3\right),$ | $\tau = \omega - 3\theta$ |

***Multiplied by***

$+\cos(\tau - 3g' - 3\omega').$

### *Type* $a[\Omega]_{10}$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{15}{2}(\gamma^3 - \gamma^5)(\gamma' - 5\gamma'^3 + 5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega$ |
| $+\frac{15}{8}(\gamma - \gamma^3)^2(6\gamma'^2 - 20\gamma'^4 + 15\gamma'^6)\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega + \theta$ |
| $+\frac{15}{8}(\gamma^4 - \gamma^6)(1-\gamma'^2)(1 - 10\gamma'^2 + 15\gamma'^4)\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega - \theta$ |
| $-\frac{15}{4}(\gamma - 2\gamma^3 + \gamma^5)(2\gamma'^3 - 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega + 2\theta$ |
| $+\frac{15}{4}\gamma^5(\gamma' - 4\gamma'^3 + 3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega - 2\theta$ |
| $+\frac{15}{8}(1-\gamma^2)^3(\gamma'^4 - \gamma'^6)\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega + 3\theta$ |
| $+\frac{15}{8}\gamma^6(\gamma' - \gamma'^3)^2\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau = 3\omega - 3\theta$ |

***Multiplied by***

$+\cos(\tau + g' + \omega').$

### *Type* a[Ω]$_{11}$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{15}{2}(\gamma^3-\gamma^5)(\gamma'-5\gamma'^3+5\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega$ |
| $+\frac{15}{8}(\gamma-\gamma^3)^2(1-\gamma'^2)(1-10\gamma'^2+15\gamma'^4)\cdot\left(-\frac{35}{8}e^3\right)$ | $\tau=3\omega+\theta$ |
| $+\frac{15}{8}(\gamma^4-\gamma^6)(6\gamma'^2-20\gamma'^4+15\gamma'^6)\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega-\theta$ |
| $-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(\gamma'-4\gamma'^3+3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega+2\theta$ |
| $+\frac{15}{4}\gamma^5(2\gamma'^3-3\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega-2\theta$ |
| $+\frac{15}{8}(1-\gamma^2)^3(\gamma'-\gamma'^3)^2\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega+3\theta$ |
| $+\frac{15}{8}\gamma^6(\gamma'^4-\gamma'^6)\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega-3\theta$ |

***Multiplied by***

$+\cos(\tau-g'-\omega').$

### *Type* a[Ω]$_{12}$

| *Factors* | *cos terms* |
|---|---|
| $-\frac{25}{2}(\gamma^3-\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega$ |
| $+\frac{75}{8}(\gamma-\gamma^3)^2(\gamma'^4-\gamma'^6)\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega+\theta$ |
| $+\frac{75}{8}(\gamma^4-\gamma^6)(\gamma'-\gamma'^3)^2\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega-\theta$ |
| $-\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right)$ | $\tau=3\omega+2\theta$ |
| $-\frac{15}{4}\gamma^5(\gamma'-2\gamma'^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega-2\theta$ |
| $+\frac{5}{8}(1-\gamma^2)^3\gamma'^6\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega+3\theta$ |
| $+\frac{5}{8}\gamma^6(1-\gamma'^2)^3\cdot\left(-\frac{35}{8}e^3\right),$ | $\tau=3\omega-3\theta$ |

***Multiplied by***

$+\cos(\tau+3g'+3\omega').$

*Type* $a[\Omega]_{13}$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{25}{2}(\gamma^3-\gamma^5)(\gamma'^3-\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega$ |
| $+\frac{75}{8}(\gamma^2-2\gamma^4+\gamma^6)(\gamma'^2-2\gamma'^4+\gamma'^6)\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega+\theta$ |
| $+\frac{75}{8}(\gamma^4-\gamma^6)(\gamma'^4-\gamma'^6)\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega-\theta$ |
| $+\frac{15}{4}(\gamma-2\gamma^3+\gamma^5)(\gamma'-2\gamma'^3+\gamma'^5)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega+2\theta$ |
| $+\frac{15}{4}\gamma^5\gamma'^5\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega-2\theta$ |
| $+\frac{5}{8}(1-\gamma^2)^3(1-\gamma'^2)^3\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega+3\theta$ |
| $+\frac{5}{8}\gamma^6\gamma'^6\cdot\left(-\frac{35}{8}e^3\right)$, | $\tau = 3\omega-3\theta$ |

*Multiplied by*

$+\cos(\tau-3g'-3\omega')$ .

The standard equations for variations of elliptic elements may be used in connection with this development. In particular, for the variation of the perigee height the equation is

$$\frac{d\,\delta q}{dt} = \frac{\sqrt{1-e^2}}{na^2e}\,\frac{\partial\, a\Omega}{\partial\omega}\,.$$

The proposed scheme leads to a form of the development which is standard in celestial mechanics. It differs from Kozai's scheme only by arrangement. The following method was used to obtain the development on the automatic computing machine:

$$\begin{aligned}
G = {} & \cos^2\frac{i}{2}\;\cos^2\frac{i'}{2}\;\cos(\omega+\theta-g'-\omega') \\
& +\sin^2\frac{i}{2}\;\cos^2\frac{i'}{2}\;\cos(\omega-\theta+g'+\omega') \\
& +\cos^2\frac{i}{2}\;\sin^2\frac{i'}{2}\;\cos(\omega+\theta+g'+\omega') \\
& +\sin^2\frac{i}{2}\;\sin^2\frac{i'}{2}\;\cos(\omega-\theta-g'-\omega') \\
& +\frac{1}{2}\sin i\cdot\sin i'\;\cos(\omega-g'-\omega') \\
& -\frac{1}{2}\sin i\cdot\sin i'\;\cos(\omega+g'+\omega')\,.
\end{aligned}$$

Then,

$$H = G + 2 \frac{\partial G}{\partial g'} e' \sin g' = G + \frac{\partial G}{\partial g'} (f' - g') ,$$

$$K = \left(\frac{3}{2} H^2 - \frac{1}{2}\right)(1 + 3e' \cos g') + \left(\frac{5}{2} H^3 - \frac{3}{2} H\right) (1 + 4e' \cos g')$$

$$= \left(\frac{3}{2} H^2 - \frac{1}{2}\right) \left(\frac{a'}{r'}\right)^3 + \left(\frac{5}{2} H^3 - \frac{3}{2} H\right) \left(\frac{a'}{r'}\right)^4 .$$

The terms of $K$ are then sorted into four groups according to the coefficient of $\omega$ in their arguments: the $2\omega$ terms are multiplied by $\left(+ \frac{5}{2} e^2\right)$; the $0 \cdot \omega$ terms are multiplied by $\left(1 + \frac{3}{2} e^2\right)$; the $1 \cdot \omega$ terms are multiplied by $\left(- \frac{5}{2} e - \frac{15}{8} e^3\right)$; and the $3\omega$ terms are multiplied by $\left(- \frac{35}{8} e^3\right)$.

In order to include the main effect of the lunar and solar perturbations in the method used at the NASA Space Computing Center (Reference 6), the development of the disturbing function must be obtained in terms of the eccentric anomaly $\xi$ of the satellite. By keeping the second harmonic only and using the formula

$$\left(\frac{r}{a}\right)^2 \cos (2f + 2\tau) = (1 + \beta^2)^{-2} \Big\{ \beta^4 \cos (- 2\xi + 2\tau)$$

$$- 4\beta^3 \cos (- \xi + 2\tau) + 6\beta^2 \cos 2\tau$$

$$- 4\beta \cos (+ \xi + 2\tau) + \cos (+ 2\xi + 2\tau) \Big\} ,$$

the following computational scheme is obtained. This scheme is valid for all eccentricities and all inclinations. The common factor $(m'a^3/a'^3)(1 + \beta^2)^{-2}$ is omitted.

*Type* $a\Omega_1$

| *Factors* | | *cos terms* |
|---|---|---|
| $+\frac{1}{4}(1 - 6\gamma^2 + 6\gamma^4)(1 - 6\gamma'^2 + 6\gamma'^4)$, | | $2\tau = 0$ |
| $+ 3(\gamma^2 - \gamma^4)(\gamma'^2 - \gamma'^4)$, | | $2\tau = 2\theta$ |
| $+ 3(\gamma - 2\gamma^3)(\gamma' - 2\gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}$ | | $2\tau = \theta$ |
| | ............ | ............ |
| | $+\frac{3}{2} e' + \cdots$ | $- g'$ |
| | $(1 - e'^2)^{-\frac{3}{2}}$ | $0 . g'$ |
| | $+\frac{3}{2} e'$ | $+ g'$ |
| | ............ | ............ |
| $+ \beta^2$ | | $- 2\xi + 2\tau$ |
| $- 2\beta - 2\beta^3$ | | $- \xi + 2\tau$ |
| $1 + 4\beta^2 + \beta^4$ | | $0 . \xi + 2\tau$ |
| $- 2\beta - 2\beta^3$ | | $+ \xi + 2\tau$ |
| $+ \beta^2$ | | $+ 2\xi + 2\tau$ |

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## *Type* $a\Omega_2$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma^2-\gamma^4)(\gamma'^2-\gamma'^4)$, | $2\tau = 2\omega$ |
| $+\frac{3}{4}(1-\gamma^2)^2(1-\gamma'^2)^2$, | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{4}\gamma^4\gamma'^4$, | $2\tau = 2\omega - 2\theta$ |
| $+3(\gamma-\gamma^3)(\gamma'-\gamma'^3)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$, | $2\tau = 2\omega + \theta$ |
| $+3\gamma^3\gamma'^3\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$, | $2\tau = 2\omega - \theta$ |

| | | |
|---|---|---|
| | ............... | ............... |
| | $+\frac{7}{2}e' + \cdots$ | $-3g' - 2\omega'$ |
| | $1-\frac{5}{2}e'^2 + \cdots$ | $-2g' - 2\omega'$ |
| | $-\frac{1}{2}e' + \cdots$ | $-g' - 2\omega'$ |
| | ............... | ............... |
| $+\beta^4$ | | $-2\xi + 2\tau$ |
| $-4\beta^3$ | | $-\xi + 2\tau$ |
| $+6\beta^2$ | | $0.\xi + 2\tau$ |
| $-4\beta$ | | $+\xi + 2\tau$ |
| $+1$ | | $+2\xi + 2\tau$ |

## *Type* $a\Omega_3$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{3}{2}(\gamma^2 - \gamma^4)(1 - 6\gamma'^2 + 6\gamma'^4)$, | $2\tau = 2\omega$ |
| $+\frac{3}{2}(1 - \gamma^2)^2(\gamma'^2 - \gamma'^4)$, | $2\tau = 2\omega + 2\theta$ |
| $+\frac{3}{2}\gamma^4(\gamma'^2 - \gamma'^4)$, | $2\tau = 2\omega - 2\theta$ |
| $-3(\gamma - \gamma^3)(\gamma' - 2\gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}$, | $2\tau = 2\omega + \theta$ |
| $+3\gamma^3(\gamma' - 2\gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}$, | $2\tau = 2\omega - \theta$ |

| | | |
|---|---|---|
| | ............... | ............... |
| | $+\frac{3}{2}e' + \cdots$ | $-g'$ |
| | $1 + \frac{3}{2}e'^2 + \cdots$ | $0.g'$ |
| | $+\frac{3}{2}e' + \cdots$ | $+g'$ |
| | ............... | ............... |
| $+\beta^4$ | | $-2\xi + 2\tau$ |
| $-4\beta^3$ | | $-\xi + 2\tau$ |
| $+6\beta^2$ | | $0.\xi + 2\tau$ |
| $-4\beta$ | | $+\xi + 2\tau$ |
| $+1$ | | $+2\xi + 2\tau$ |

## *Type* $a\Omega_4$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{3}{2}(1 - 6\gamma^2 + 6\gamma^4)(\gamma'^2 - \gamma'^4)$, | $2\tau = 2\omega'$ |
| $+\frac{3}{2}(\gamma^2 - \gamma^4)\gamma'^4$, | $2\tau = 2\omega' + 2\theta$ |
| $+\frac{3}{2}(\gamma^2 - \gamma^4)(1 - \gamma'^2)^2$, | $2\tau = 2\omega' - 2\theta$ |
| $+ 3(\gamma - 2\gamma^3)\gamma'^3\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}$, | $2\tau = 2\omega' + \theta$ |
| $- 3(\gamma - 2\gamma^3)(\gamma' - \gamma'^3)\sqrt{1 - \gamma^2}\sqrt{1 - \gamma'^2}$, | $2\tau = 2\omega' - \theta$ |

| | | |
|---|---|---|
| | ............... | ............... |
| | 0 *(exact)* | $0.g'$ |
| | $-\frac{1}{2}e' + \cdots$ | $+ g'$ |
| | $1 - \frac{5}{2}e'^2 + \cdots$ | $+ 2g'$ |
| | $+\frac{7}{2}e' + \cdots$ | $+ 3g'$ |
| | ............... | ............... |
| $+ \beta^2$ | | $- 2\xi + 2\tau$ |
| $- 2\beta - 2\beta^3$ | | $- \xi + 2\tau$ |
| $1 + 4\beta^2 + \beta^4$ | | $0.\xi + 2\tau$ |
| $- 2\beta - 2\beta^3$ | | $+ \xi + 2\tau$ |
| $+ \beta^2$ | | $+ 2\xi + 2\tau$ |

## *Type* $a\Omega_5$

| *Factors* | *cos terms* |
|---|---|
| $+\frac{9}{2}(\gamma^2-\gamma^4)(\gamma'^2-\gamma'^4)$, | $2\tau = 2\omega + 2\omega'$ |
| $+\frac{3}{4}(1-\gamma^2)^2\gamma'^4$, | $2\tau = 2\omega + 2\theta + 2\omega'$ |
| $+\frac{3}{4}\gamma^4(1-\gamma'^2)^2$, | $2\tau = 2\omega - 2\theta + 2\omega'$ |
| $-3(\gamma-\gamma^3)\gamma'^3\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$, | $2\tau = 2\omega + \theta + 2\omega'$ |
| $-3\gamma^3(\gamma'-\gamma'^3)\sqrt{1-\gamma^2}\sqrt{1-\gamma'^2}$, | $2\tau = 2\omega - \theta + 2\omega'$ |

| | | |
|---|---|---|
| | ............... | ............... |
| | 0 *(exact)* | $0.g'$ |
| | $-\frac{1}{2}e' + \cdots$ | $+g'$ |
| | $1-\frac{5}{2}e'^2 + \cdots$ | $+2g'$ |
| | $+\frac{7}{2}e' + \cdots$ | $+3g'$ |
| | ............... | . ............. |
| $+\beta^4$ | | $-2\xi + 2\tau$ |
| $-4\beta^3$ | | $-\xi + 2\tau$ |
| $+6\beta^2$ | | $0.\xi + 2\tau$ |
| $-4\beta$ | | $+\xi + 2\tau$ |
| $+1$ | | $+2\xi + 2\tau$ |

## THE INTEGRATION PROBLEM

The method used at the NASA Space Computing Center is based on Hansen's idea of separating the perturbations *in* the orbit plane from the perturbations *of* the orbit plane. A typical differential equation of the method takes the form

$$\frac{dQ}{dE} = \sum A \begin{matrix}\cos\\ \sin\end{matrix} \alpha ,$$

$$\alpha = iE + j\omega + k\Omega + i'g' + j'\omega' + k'\Omega' + pF ,$$

where $F$ is considered as a constant and

$$\omega = \omega_0 + \omega_1 (E - E_0) , \qquad g' = g_0' + n' (t - t_0) ,$$

$$\Omega = \Omega_0 + \Omega_1 (E - E_0) , \qquad \omega' = \omega_0' + n'\omega_1' (t - t_0) ,$$

$$\Omega' = \Omega_0' + n'\Omega_1' (t - t_0) .$$

The time $t$ can be eliminated by means of Kepler's equation,

$$E - e \sin E = g = g_0 + n(t - t_0) ;$$

and, after several transformations involving Bessel functions, the right-hand side can be reduced to an integrable form. However, instead of this complicated method, the integration can be performed in such a way that $E$ and $g'$ are both kept in the argument. A similar method was used by Hansen (Reference 7) in his development of the perturbations of Encke's comet by Saturn.

From Kepler's equation and from

$$g' = g_0' + n' (t - t_0)$$

it can be deduced that

$$\frac{dg'}{d\xi} = m(1 - e \cos \xi) \qquad \left(\text{where } m = \frac{n'}{n}\right) ,$$

and

$$\frac{dQ}{d\xi} = \frac{\partial Q}{\partial \xi} + \frac{\partial Q}{\partial g'} \cdot \frac{dg'}{dt} = \frac{\partial Q}{\partial \xi} + m \frac{\partial Q}{\partial g'} (1 - e \cos \xi) .$$

Thus the problem is reduced to the integration of the linear partial differential equation

$$\frac{\partial Q}{\partial \xi} + m(1 - e \cos \xi) \frac{\partial Q}{\partial g'} = \sum A \begin{matrix}\cos\\ \sin\end{matrix} \alpha = X .$$

extended their calculations to the case of the paddle-wheel satellite, Explorer VI

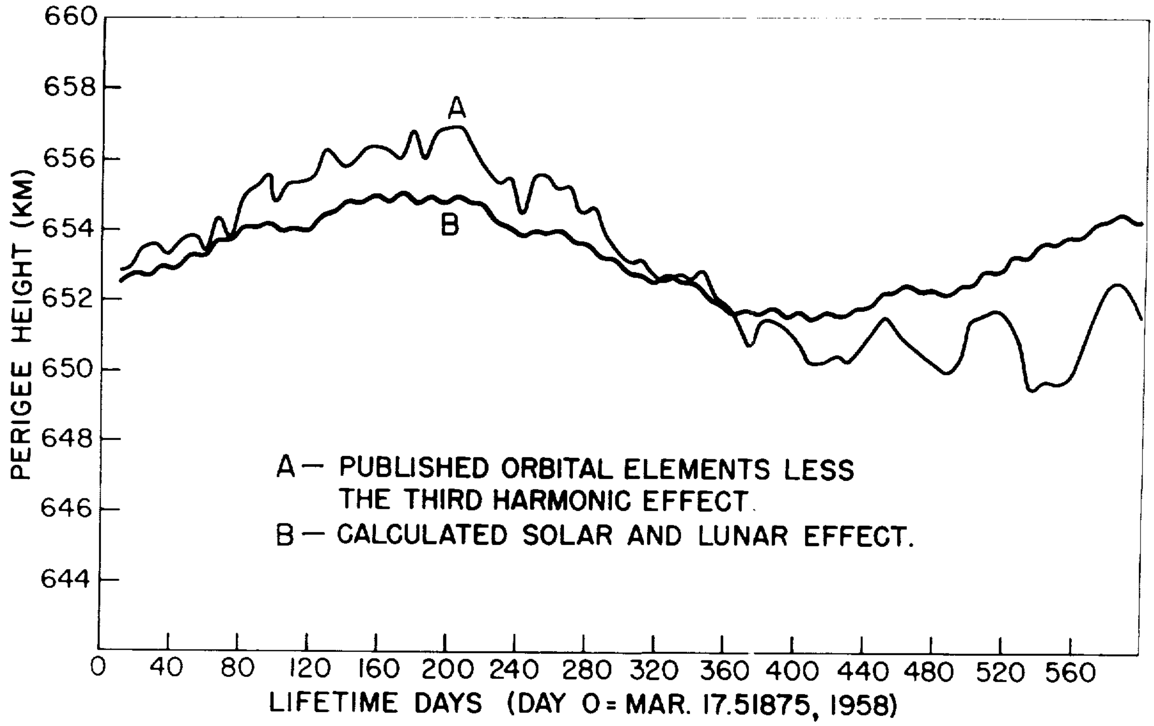


Figure 1 - Solar and lunar perturbations on the perigee height of Vanguard I (1958 $\beta_2$)

(Reference 1). Explorer VI has an apogee of 48,700 kilometers, a perigee of 6640 kilometers, and an orbital inclination of 47.3 degrees; it was found that this highly eccentric orbit produces substantial lunar and solar perturbations which decrease the perigee altitude rapidly and shorten the orbital lifetime from several decades to a probable value of two years.

The interesting work of Kozai and Whitney encouraged further exploration of the possible lunar and solar effects on perigee height for satellite orbits of large eccentricity. In general, both the eccentricity and the perigee height vary with time as a result of these effects. The amplitudes, frequencies, and relative phases of the variations are determined by the orbit parameters, among which the hour of launch is of considerable importance. *For a special set of launch conditions, and for representative orbit parameters, the perigee height may be made to rise steadily over the course of several years at a rate of 1 kilometer per day. Thus, the sun and the moon may provide a substantial perigee boost for the satellite under properly chosen circumstances. For other conditions the perturbations may be minimized to obtain a relatively stable orbit.* These considerations may be of importance in deciding the launch programs for future satellites with highly eccentric orbits.

From the perturbing function developed herein, the rate of change of perigee height was found by the method of variation of constants. Letting q be the perigee height,

$$\frac{dq}{dt} = -\frac{(1-e)^2}{nae}\frac{\partial\Omega}{\partial\sigma} + \frac{\sqrt{1-e^2}}{nae}\frac{\partial\Omega}{\partial\omega}$$

where $\sigma$ is essentially the mean anomaly g of the satellite, $\omega$ is the argument of perigee of the satellite, and $\Omega$ is the perturbing function.

The first-order perturbations of the gravitational effect of the sun and moon were developed, and several assumptions which considerably simplified the expression for the potential were made. It was assumed that the moon's orbital plane is coincident with the

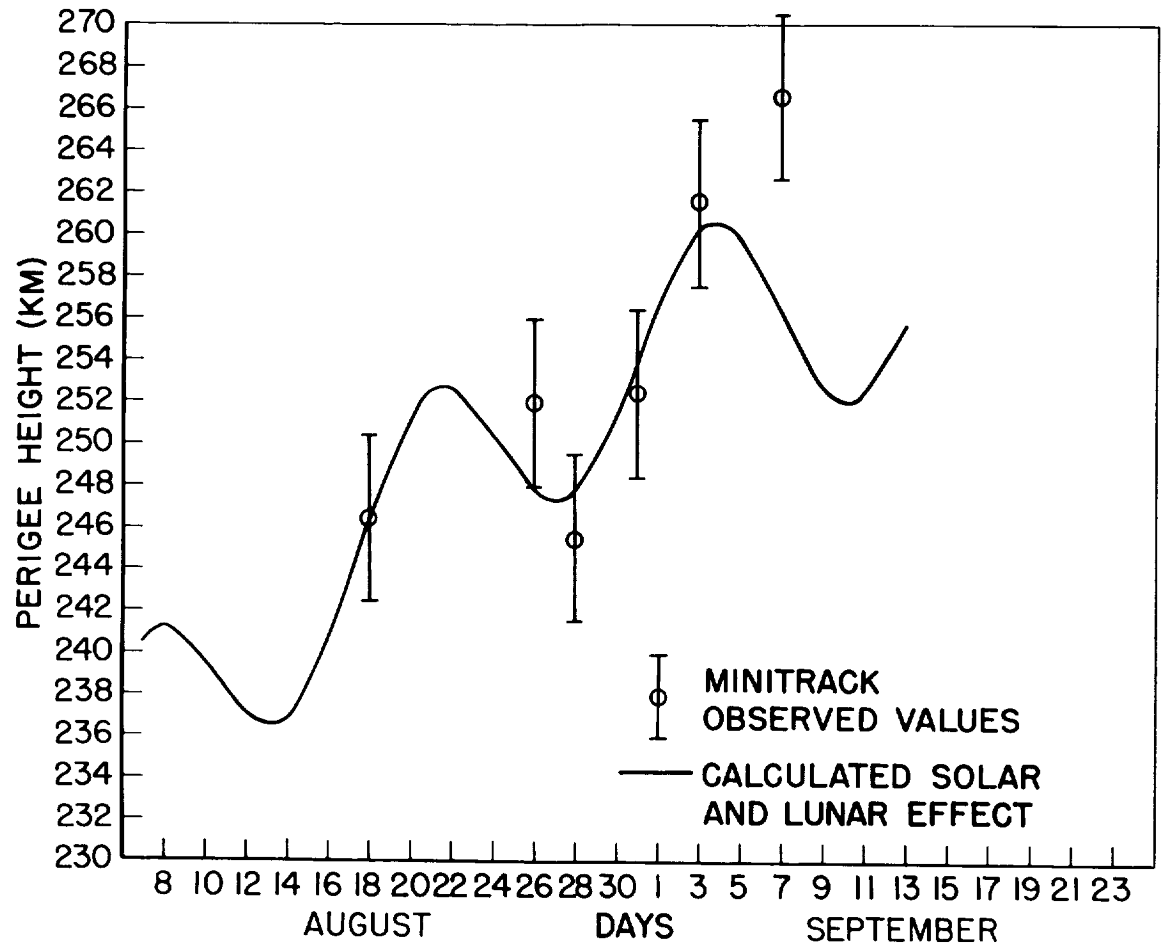


Figure 2 - Solar and lunar perturbations on the perigee height of Explorer VI (1959 $\delta$)

sun's. Only the long-duration effects on perigee altitude were considered; this eliminated terms with periods of the mean anomaly (and fractional parts thereof) of the satellite and, for the most part, of the moon (i.e., $g$ and $g'_{\mathbb{C}}$).

Hence, the expression $dq/dt$ becomes

$$\frac{dq}{dt} = + \frac{\sqrt{1 - e^2}}{nae} \frac{\partial \Omega}{\partial \omega} ,$$

which is a trigonometric series whose arguments depend on various combinations of $2\omega$, $\theta$, $\lambda_{\odot}$, $\lambda_{\mathbb{C}}$, $\omega_{\odot}$, $\omega_{\mathbb{C}}$. In terms of these elements several possible resonance effects can be recognized, two of which are associated with the following conditions:

$$2\dot{\omega} - 2(\dot{\lambda}_{\odot} - \dot{\theta}) = 0 , \tag{10a}$$

$$2\dot{\omega} + 2(\dot{\lambda}_{\odot} - \dot{\theta}) = 0 , \tag{10b}$$

where $\dot{\omega}$, $\dot{\lambda}$, $\dot{\theta}$ are the average angular velocities of $\omega$, $\lambda$, $\theta$, respectively.

These resonance conditions have a simple interpretation. For example, in Equation 10a, $\lambda_{\odot} - \theta$ represents the longitude of the sun relative to the line of nodes; and $\omega$, the position of the perigee in the orbital plane, is also defined relative to the line of nodes. Therefore, in a system in which the line of nodes is fixed, the satisfaction of the resonance condition (case 10a) signifies that the mean angular velocities of the sun and perigee are equal; that is, the line of apsides follows the sun. In this case the orbital perturbations produced by the sun are clearly maximized. In Equation 10b, the sun and the line of apsides have the same period of revolution but in opposite directions. Again, it is clear that the solar perturbations will be maximized.

It was found that the effects on perigee height may be maximized or minimized by choosing suitable values of the orbital inclination and the time of launch. A long-period effect occurs when the inclination to the equator is near 63.4 degrees – the critical angle at which the motion of the argument of perigee is small. There is a $2\omega$ term in $dq/dt$ which at this inclination produces a near-resonance effect, causing the perigee height to change almost secularly. For an orbit with an apogee of 46,550 kilometers and a perigee of 6650 kilometers, the rate of change of perigee is about 1 kilometer per day, as shown in Figure 3. The precise magnitude of the rate of change depends on the initial argument of perigee. The hour of launch does not affect this result.

At angles of inclination other than 63.4 degrees, a variety of effects may be obtained by a suitable choice of the hour of launch. Selecting the hour of launch is equivalent to selecting the longitude of the ascending node, with any value available once in 24 sidereal hours. The results of three different launch times are shown in Figure 4, employing the same apogee and perigee as in Figure 3 with an argument of perigee equal

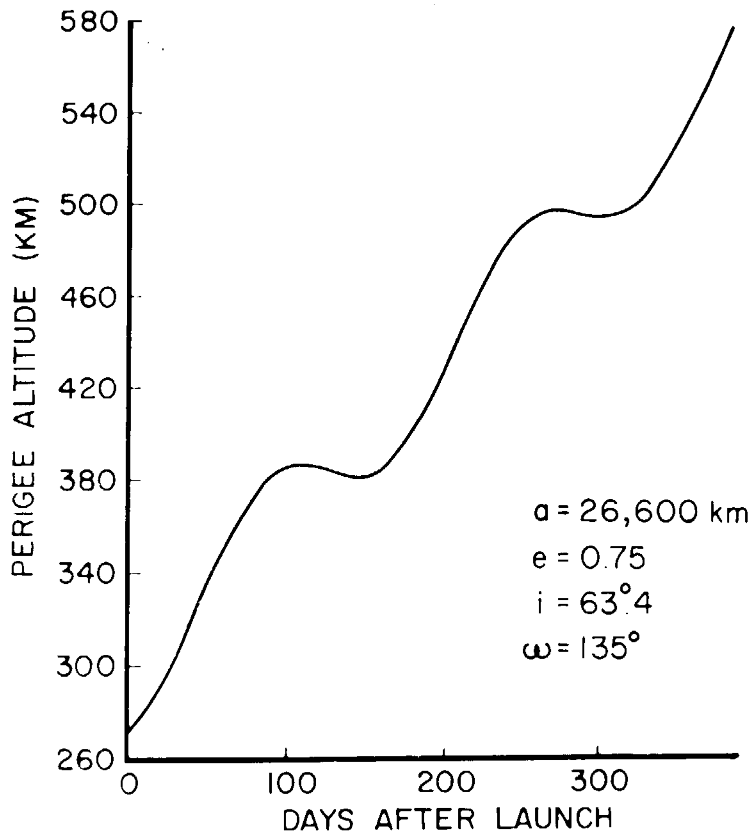


Figure 3 - Rate of change of perigee for orbital inclination near 63.4 degrees

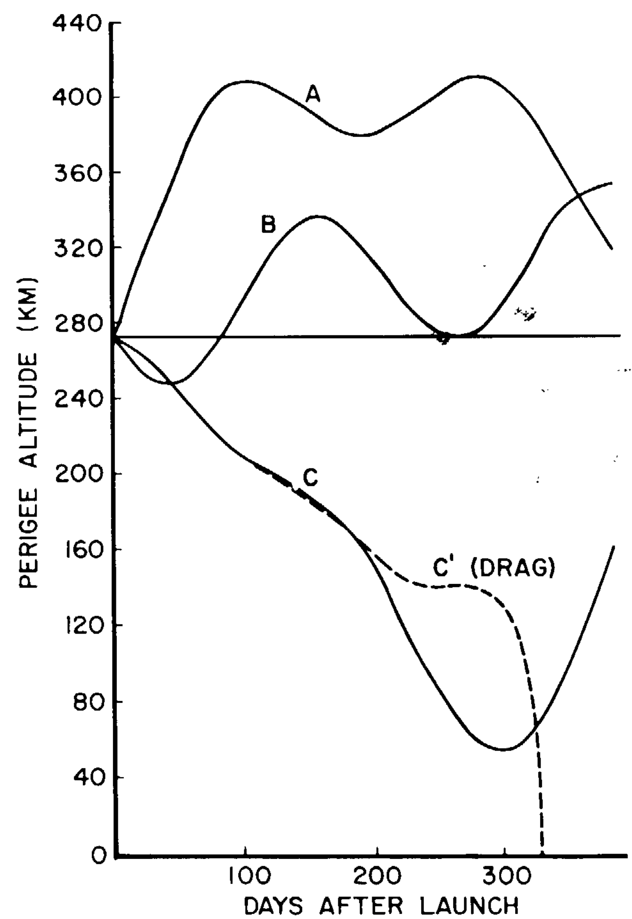


Figure 4 - Results of three different choices of launch time on February 1, 1960: (A) $7^h$ U.T., (B) $23^h$ U.T., (C) $13^h$ U.T.; (C') shows the addition of drag to (C)

to 135 degrees and an equatorial inclination of 28 degrees for February 1, 1960. Curve A corresponds to a launch time of 7 hours U.T. on February 1, curve B to 23 hours U.T., and curve C to 13 hours U.T. Cases A and C show rapid initial variations of perigee height — advantageous and disadvantageous. Case B represents a relatively stable orbit.

Curve C' represents the addition of drag to case C. It rises initially above the solar and lunar perturbation curve because the drag decreases the period and the eccentricity, and these changes in turn decrease the solar and lunar perturbations. It is interesting to note that the lifetime for a satellite with the parameters of Figure 4 is 25 years in the absence of lunar and solar perturbations and approximately 1 year with these perturbations.

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